A Survey of Tools Detecting the Dynamical Properties of One-Dimensional Families

Marek LAMPART[1,2], Tomas MARTINOVIC[1]

[1]IT4Innovations National Supercomputing Center, VSB–Technical University of Ostrava,
17. listopadu 15/2172, 708 33 Ostrava, Czech Republic
[2]Department of Applied Mathematics, Faculty of Electrical Engineering and Computer Science,
VSB–Technical University of Ostrava, 17. listopadu 15/2172, 708 33 Ostrava, Czech Republic

marek.lampart@vsb.cz, tomas.martinovic@vsb.cz

Abstract. *One-dimensional dynamical systems attract researches for more than half a century and the topic was inspired by many real problems. Mainly piecewise linear and polynomial maps were considered and researched under several motivations from different scientific fields. As a main aim of the paper, the Logistic (polynomials of the second order) and the Tent (piecewise linear maps with two pieces) families are considered and studied. The dynamical properties of both families are derived using bifurcation diagrams, Lyapunov exponents, and newly established techniques like the 0-1 test for chaos and recurrence matrices.*

Keywords

Entropy, Logistic family, Tent family, recurrence matrix, 0-1 test for chaos.

1. Introduction

Many phenomena coming from various scientific disciplines are described by discrete dynamical system

$$x_{n+1} = f(x_n),\ n = 0, 1, 2, \ldots, \tag{1}$$

where points x_n belong to a compact metric space $\mathbb{X}$ and $f : \mathbb{X} \to \mathbb{X}$ is a continuous transformation. The main aim of the theory of discrete dynamical systems is centred on the understanding how the trajectories of all points from $\mathbb{X}$ look like.

Classical discrete dynamical systems have been highly considered in the literature (see e.g. [1] or [2]) because they are good examples of problems coming from the theory of topological dynamics and model numerous phenomena from biology, physics, chemistry, engineering and social sciences (see, for example, [3], [4], [5] or [6] and references therein). In most formulations of such models, f is a C^∞, analytical, or a polynomial map.

The dynamics of the second order polynomial (the Logistic map Eq. (2) for $\mu = 1$) was derived and researched in [7]. Later on, more practical examples in one-dimensional dynamical systems were deduced under several real motivations, see e.g. [8], and many interesting results on piecewise linear maps were reached [9] and [10].

The theory of one-dimensional dynamical systems was deeply researched in the past decades, see e.g. [11] and [12]. The following classification of the periodic structure can be considered to be the crucial result of understanding one-dimensional dynamics [13]. Here, the point $x \in X$ is *fixed* if $f(x) = x$ and is *periodic* with period p if $f^p(x) = x$ and $f^r(x) \neq x$ for any $r = 1, 2, \ldots p - 1$. Here, $f^n(x) = f(f^{n-1}(x))$ stands for n fold composition of f, and denoting $x_n = f^n(x_0)$ the n-th iteration of x_0 under f for simplicity.

Theorem 1 (Sharkovskii, 1964)**.** *Let f be a continuous map of the unit closed interval I with a periodic point of the period p, and let:*

$$p \prec q.$$

Then f has a periodic point of period q.
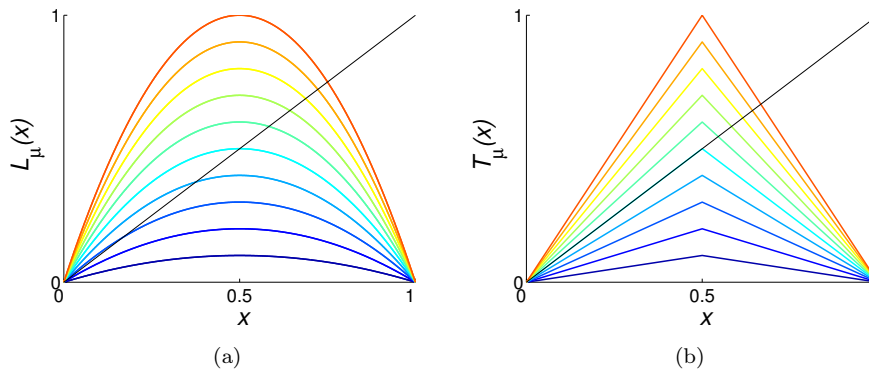
Fig. 1: Graphs of the Logistic L_μ (a) and Tent (b) T_μ maps for the value of parameter μ form 0.1 up to 1 by 0.1.

The ordering $\prec$, used in the above stated theorem, of the set of positive integers $\mathbb{N}$ is defined in the following way:

$$3 \prec 5 \prec 7 \prec 9 \prec \ldots$$
$$\prec \quad 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec 2 \cdot 9 \prec \ldots$$
$$\prec \quad 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec 2^2 \cdot 7 \prec 2^2 \cdot 9 \prec \ldots$$
$$\prec \quad 2^3 \cdot 3 \prec 2^3 \cdot 5 \prec 2^3 \cdot 7 \prec 2^3 \cdot 9 \prec \ldots$$
$$\prec \quad \ldots$$
$$\prec \quad 2^5 \prec 2^4 \prec 2^3 \prec 2^2 \prec 2 \prec 1$$

That means if $f : I \to I$ has periodic point of period 3, then it has periods of all orders.

Remark 1. *It is worthy to note that it is not possible to generalize the above stated theorem directly. For example, rational rotation on the unit circle does not hit the conclusion of Thm. 1. But some generalizations are possible, for more, see e.g. [14].*

The above stated theorem opens the way for very rich dynamical properties, and together with sensitive dependence on initial conditions [15] the feeling of "chaos" can be reached. The first notion of chaos was given by Li and Yorke [16] and is defined as follows.

A set $S \subset X$ containing at least two points is called an *LY-scrambled set for f* if for any two $x \neq y$ in S is

$$\limsup_{n \to \infty} d(f^n(x), f^n(y)) > 0 \quad \text{and}$$

$$\liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0.$$

The map f is *Li and Yorke chaotic* if there is an uncountable LY-scrambled set [16].

Theorem 2 (Li and Yorke, 1975)**.** *Let f be a continuous map of the unit closed interval I with a periodic point of the period 3, then f is Li and Yorke chaotic.*

It was proved for interval (respectively circle) maps that the existence of a Li–Yorke pair (scrambled set contains only two points) implies the existence of an uncountable scrambled set [17] (resp. [18]). In general, Li–Yorke chaos is not implied by the presence of a Li–Yorke pair, see [19], for instance.

Remark 2. *The notion of chaos is widely studied and many different definitions appear as well as numerous comparative papers have been written, see e.g. [20], [21] and [22] and references therein.*

As examples of continuous maps of the unit interval $I = [0, 1]$ the Logistic and Tent maps are considered. We introduce the family of Logistic maps in dependence on parameter $\mu \in [0, 1]$:

$$L_\mu(x) = \mu 4x(1 - x), \tag{2}$$

and the Tent family also in dependence on parameter $\mu \in [0, 1]$:

$$T_\mu(x) = \mu(1 - |2x - 1|). \tag{3}$$

For the special case of parameter $\mu = 1$, $T_1(x)$ and $L_1(x)$ are topologically conjugated meaning that is there is a homeomorphism $h : I \to I$ such that $h \circ T_1 = L_1 \circ h$, meaning that all dynamical properties are preserved. Thus, $T_1(x)$ and $L_1(x)$ have the same dynamical properties. This is not true for many remaining parameters μ, see bifurcation diagram in Fig. 2.

On the other hand, maps $T_1(x)$ and $L_1(x)$ have different distributions. In the case of $T_1(x)$, all orbits are uniformly distributed as opposed to those of $L_1(x)$ (see e.g. [23]). See Fig. 3 for comparison. In both cases, histogram of 500 000 iterations of the initial value $x_0 = \sqrt{2}/2$ were done.

That opens natural questions on comparison since from the practical point of view implementation and iteration of L_μ is much simpler than T_μ, but the advantage of T_μ versus L_μ is that the map is piecewise linear.

Hence, the understanding of the complexity of the maps plays a crucial role. Therefore, the main aim of
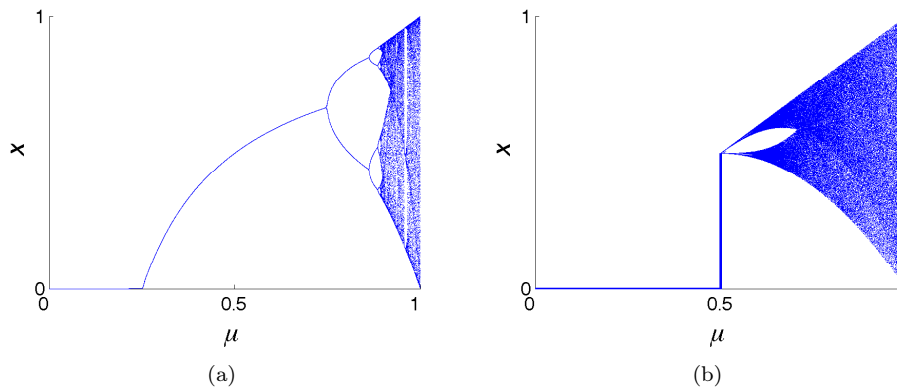
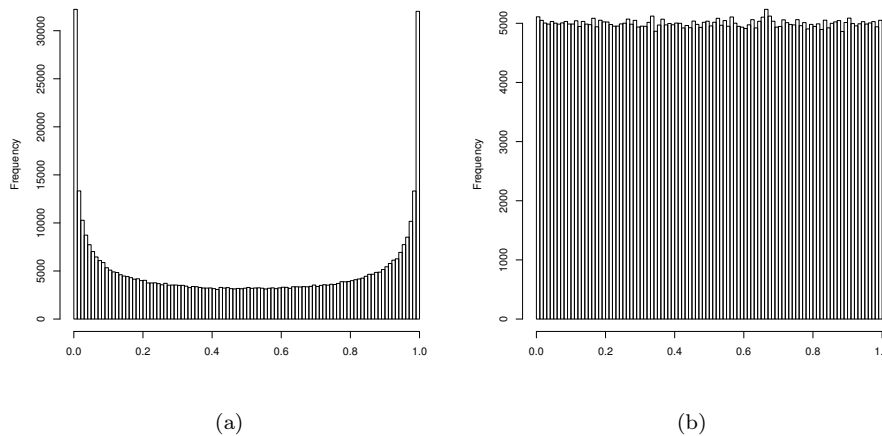**Fig. 2:** Bifurcation diagrams: (a) for the Logistic family L_μ, (b) for the Tent family T_μ.

Fig. 3: Histograms of L_1 (a) and Tent (b) T_1 maps for $x_0 = \sqrt{2}/2$ and $500\,000$ iterations.

this paper is the study of dynamical properties of two one-dimensional families, namely the Logistic Eq. (2) and Tent Eq. (3) families. For this purpose, the topological entropy in Sec. 2. , Lyapunov exponents in Sec. 3. , 0-1 test for chaos in Sec. 4. , and recurrence matrices in Sec. 5. are introduced and used.

2. Topological Entropy

An attempt to measure the complexity of a dynamical system is based on a computation of how many points are necessary in order to approximate (in some sense) with their all possible orbits of the system. A formalization of this intuition leads to the notion of topological entropy of the map f, which is due to Adler, Konheim and McAndrew [24]. We recall here the equivalent definition formulated by Bowen [25], and independently by Dinaburg [26]: the *topological entropy* of a map f is a number $\mathbf{h}(f) \in [0, \infty]$ defined by:

$$\mathbf{h}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \# E(n, f, \varepsilon),$$

where $E(n, f, \varepsilon)$ is a (n, f, ε)–span with minimal possible number of points, i.e., a set such that for any $x \in \mathbb{X}$ there is $y \in E(n, f, \varepsilon)$ satisfying $d(f^j(x), f^j(y)) < \varepsilon$ for $1 \le j \le n$.

A map f is *topologically chaotic* if its topological entropy $\mathbf{h}(f)$ is positive. In the one-dimensional case the theory was deeply researched in the past decades, see e.g. [27], [28] and [29].

The topological entropy of the Tent family is well known, see e.g. [30]:

Theorem 3 (Misiurewicz and Szlenk, 1980)**.** *Let T_μ be the Tent family. Then*

$$\mathbf{h}(T_\mu) = \begin{cases} 0, & \text{if } \mu \in [0, 0.5] \\ \log 2\mu, & \text{if } \mu \in [0.5, 1] \end{cases}.$$

The entropy of the Logistic family was recalculated utilizing the above stated theorem in [31] using so called kneading sequences, an alternative algorithms were also given by [32].
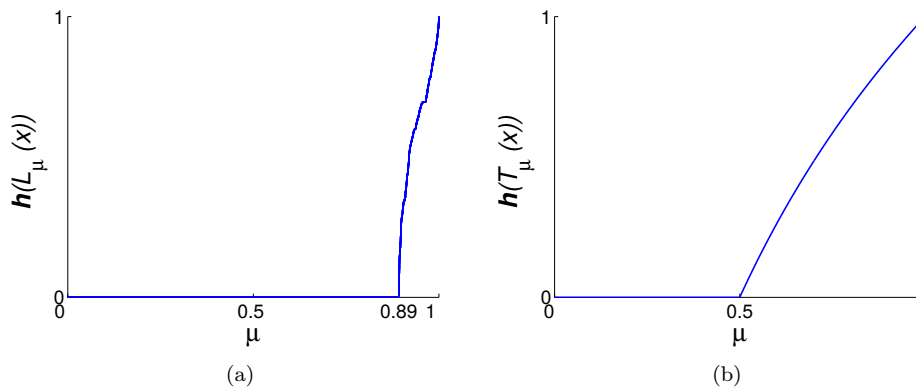
Fig. 4: Graph of the topological entropy versus parameter μ for the Logistic L_μ (a) and Tent T_μ (b) family. The x axis stands for the parameter $\mu \in [0,1]$, and the y axis is the topological entropy computed with logarithm base 2 in both cases, so $\mathbf{h}(L_\mu)$ and $\mathbf{h}(T_\mu) \in [0,1]$.

The dependence of the topological entropy of the Logistic and Tent family, respectively, is shown in Fig. 4. In the case of the Tent family it is an increasing function, while in the case of the Logistic family it is non-decreasing one with wide constant strips. The Tent map is topologically chaotic if $\mu > 0.5$, and the Logistic family is topologically chaotic if $\mu > 0.89$.

3. Lyapunov Exponent

If two infinitesimally close points of the system diverge exponentially in time and remain in the same compact space, it is considered that the system is chaotic, see e.g. [33]. The measure of this divergence is called *Lyapunov exponent*. The positive value of the Maximal Lyapunov exponent indicates the system is chaotic, for zero case the bifurcation occurs and a negative Lyapunov exponent detects regular (periodic) movement.

Now, let us consider a point x_0 and its neighboring point $x_0 + \delta$, assuming δ to be positive real number. The error err_n we did replacing the original point by its neighbor in the n-th iteration defined by:

$$\mathrm{err}_n = |f^n(x_0 + \delta) - f^n(x_0)|,$$

and the relative error by:

$$\left| \frac{\mathrm{err}_n}{\delta} \right| = \frac{|f^n(x_0 + \delta) - f^n(x_0)|}{\delta}.$$

If the map f has sensitive dependence on initial conditions (see [2]) meaning that there is ϵ such that for any $x_0 \in I$ there is $y_0 \in (x_0 - \delta, x_0 + \delta)$ and $k \in \mathbb{N}$ such that:

$$|f^k(x_0) - f^k(y_0)| \geq \epsilon,$$

we suppose the relative error to grow exponentially with n and thus:

$$
\begin{aligned}
e^{n\lambda} &= \lim_{\delta \to 0} \frac{|f^n(x_0 + \delta) - f^n(x_0)|}{\delta} = \left| \frac{\mathrm{d}}{\mathrm{d}x} f^n(x_0) \right| \\
&= |f'(x_0)f'(x_1) \ldots f'(x_{n-1})|.
\end{aligned}
$$

Hence

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)|,$$

defines the *Lyapunov exponent* of a map f with respect to the initial point x_0, if the limit exists, denoted $\mathbf{Lyap}(f(x_0))$.
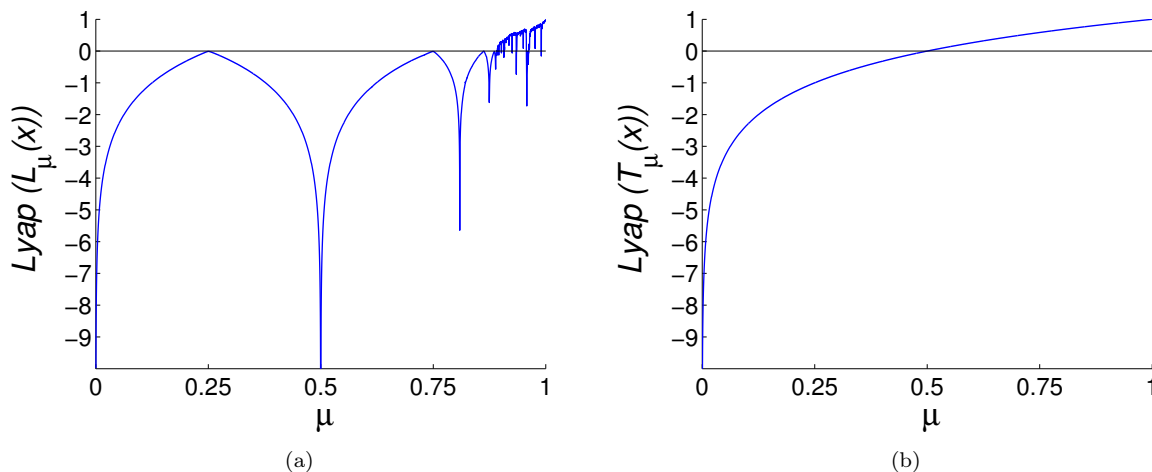
Lyapunov exponent for the Logistic family can be computed as:

$$\mathbf{Lyap}(L_\mu(x_0)) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |4\mu - 8\mu f^k(x_0)|. \quad (4)$$

In the case of the Tent family Maximum Lyapunov exponent equals to:

$$\mathbf{Lyap}(T_\mu(x_0)) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |2\mu| = \log 2\mu. \quad (5)$$

Lyapunov exponents for the Logistic and Tent families are shown in Fig. 5, in both cases $x_0 = \sqrt{2}/2$.

Inverse of the Maximal Lyapunov exponent is a number of steps which may be predicted in the system. Thus, Maximal Lyapunov exponent is directly related to predictability of the system and is of great interest in study of dynamical systems. In practice, Maximal Lyapunov exponent must be estimated from the experimental data. The generally used method is proposed in [34] and was used, for example, in [35] to estimate predictability of highway traffic speed in England.

Fig. 5: Graph of the Lyapunov exponent versus parameter μ for the Logistic L_μ (a) and Tent T_μ (b) family. The x axis stands for the parameter $\mu \in [0, 1]$, and the y axis is the topological entropy computed with logarithm base 2 in both cases, so **Lyap**(L_μ) and **Lyap**$(T_\mu) \in [0, 1]$.

4.   0-1 Test for Chaos

The 0-1 test for chaos was introduced in [36] to distinguish between regular and chaotic dynamics in deterministic dynamical systems. As an output of the test, 0 stands for regular movement and 1 for chaotic patterns. As opposed to the computational methods of the Lyapunov exponent, this method is direct on tested data, i.e. no preprocessing and only minimal computational effort is required. This method was originally stated as *regression* one, and later on in [37] it was improved as *correlation* that is faster and qualitatively gives better results; it is faster in terms of convergence. This correlation method works for a given set of observations $\phi(j)$ for $j \in \{1, 2, , 3, \ldots N\}$ as follows.

Firstly, compute the translation variables for suitable choice of $c \in (0, 2\pi)$:

$$p_c(n) = \sum_{j=1}^{N} \phi(j) \cos(jc), \quad q_c(n) = \sum_{j=1}^{N} \phi(j) \sin(jc),$$

then the mean square displacement:

$$M_c(n) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \left([p_c(j+n) - p_c(j)]^2 + [q_c(j+n) - q_c(j)]^2 \right),$$

where the limit is confident by calculating $M_c(n)$ only for $n \leq n_{cut}$ where $n_{cut} \ll N$, and put $n_{cut} = N/10$. Now, let us estimate modified mean square displacement:

$$D_c(n) = M_c(n) - \left(\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \phi(j) \right)^2 \frac{1 - \cos(nc)}{1 - \cos(c)}.$$

Put $\xi = (1, 2, \ldots n_{cut})$, $\Delta = (D_c(1), D_c(2), \ldots D_c(n_{cut}))$. Finally, we get the output of the 0-1 test as the correlation coefficient of ξ and Δ for fixed parameter c:

$$K_c = \text{corr}(\xi, \Delta) \in [-1, 1]. \qquad (6)$$

Obviously, K_c is dependent on the choice of c and as it was pointed out in [37] it is enough to get K, as the output of the 0-1 test, as the limiting value of all K_c. Our tests confirm experience of [37] that it is sufficient to introduce $K = \text{median}(K_c)$.

To avoid the resonances distorting the statistics, parameter c is chosen from the restricted interval $(\pi/5, 4\pi/5)$ for all computations, see [37]. In our tests 450 samples from 0.55 to 1.0 by 0.01 for the Tent family and 150 samples from 0.85 to 1.0 by 0.01 for the Logistic family were done. In both cases 500 000 iterations of initial value $x_0 = \sqrt{2}/2$ were performed. The output of computations is shown in Fig. 6.

5.   Recurrence Plot

Recurrence plot is a visual representation of trajectory recurrences in the dynamical system. Extensive publication concerning theory of recurrence plots and application of recurrence plots in time series analysis are [38] and [39], respectively. Generally, it is computed from the embedded time series. Embedding time series to the phase space is common practice in the nonlinear dynamical systems analysis. In the case of observing one feature of the multi-dimensional dynamical system, embedding of this observed time series should re-create phase space with same dynamics as the original system see e.g. [40].
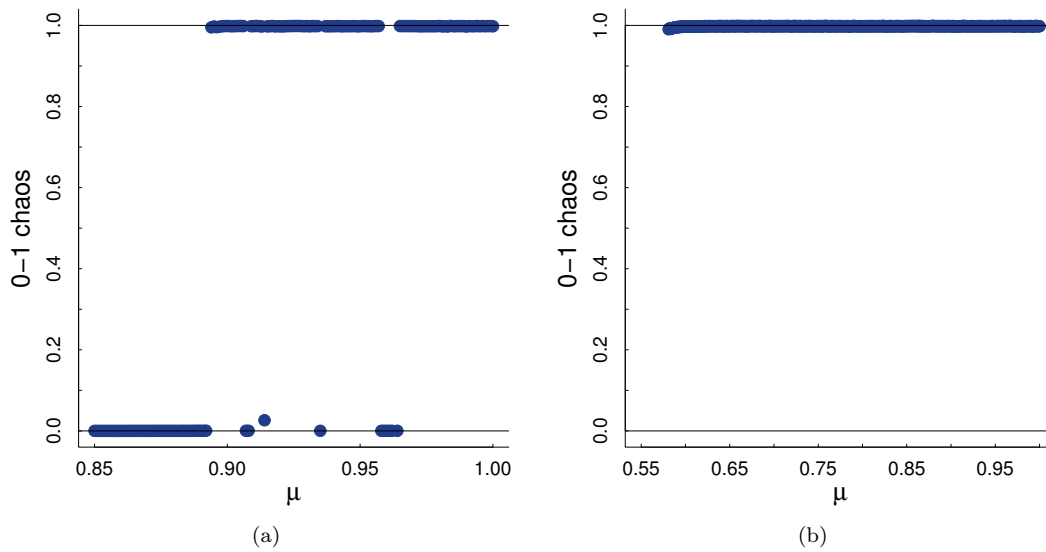
Fig. 6: Graph of the 0-1 test for chaos versus parameter μ for the Logistic L_μ (a) and Tent T_μ (b) family.

Let $\{x(t) \in \mathbb{R} \,|\, t = 1, 2, \ldots, n\}$ be observed time series of length n. Then embedded vector $X(t)$ at time t, is defined as $X(t) = [x(t), x(t+l), x(t+2l), \ldots, x(t+(m-1)l)]$, where t is the observed time, l is delay time and m is embedding dimension.

First, the distance matrix of all the vectors $X(t)$ is computed. The euclidean distance matrix is computed as:

$$D(t_1, t_2) = d(X(t_1), X(t_2)) = \qquad (7)$$

$$= \sqrt{\sum_{k=0}^{m-1} (x(t_1 + kl) + x(t_2 + kl))^2},$$

for all the pairs $X(t_1), X(t_2)$, where $t_1, t_2 \in \{1, 2, \ldots, n - (m-1)l\}$. Recurrence plot is then com-

puted as

$$RP^\varepsilon(i,j) = \begin{cases} 1, & D(i,j) < \varepsilon, \\ 0, & \text{otherwise,} \end{cases} \qquad (8)$$

for $i, j \in \{1, 2, \ldots, n - (m-1)l\}$. This may be alternatively written as $RP^\varepsilon(i,j) = \Theta(D(i,j))$, where $\Theta(\cdot)$ is the Heaviside function.

A recurrence plots for different values of parameter μ for the Logistic family and Tent family are shown in Fig. 7 and Fig. 8 respectively. The difference in dynamics is clear between the recurrence plots for Logistic map with parameter $\mu = 0.91$ and parameter $\mu = 1$. While there are many structures with large number of smaller blocks changing almost regularly in the first case. These structures are more complicated for the latter case, where the block size is more diverse and is
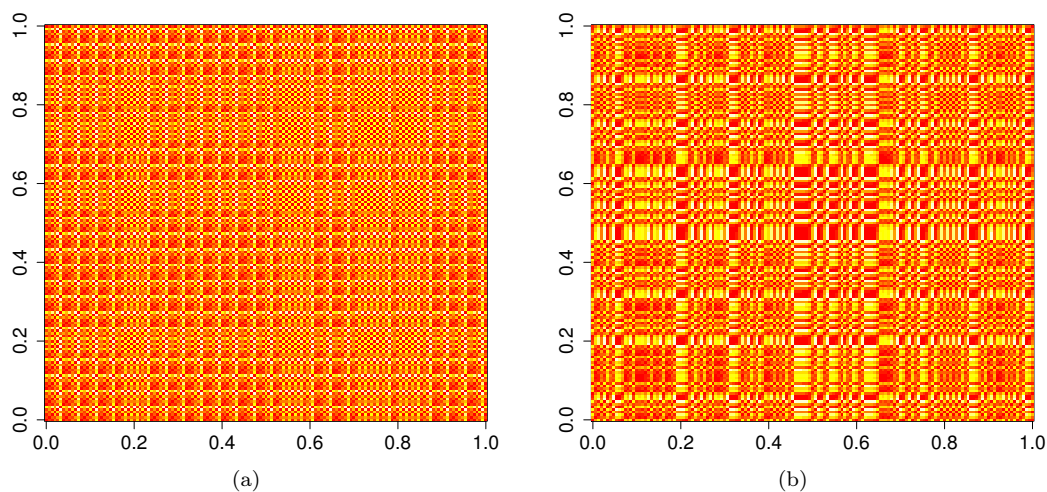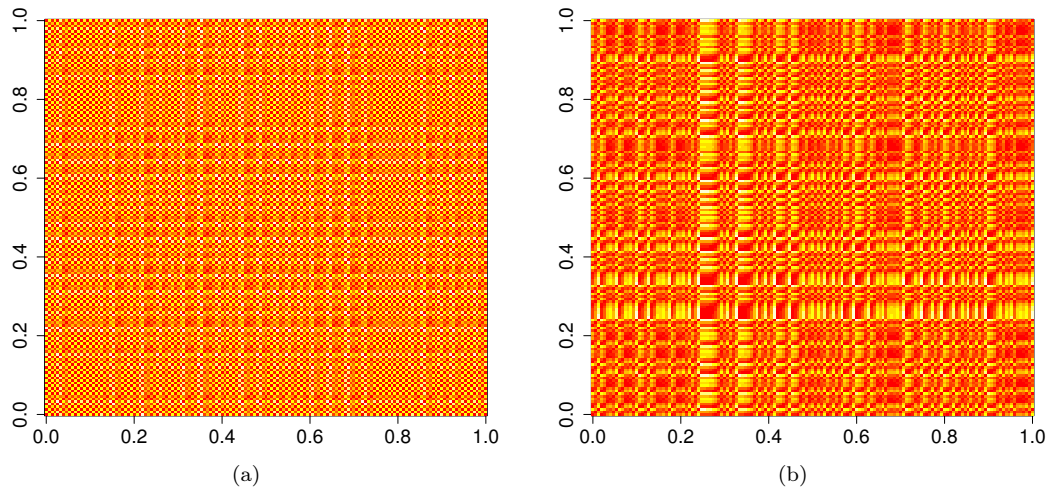

Fig. 7: Recurrence plot for the Logistic family for parameter $\mu = 0.91$ (a) and $\mu = 1$ (b).

Fig. 8: Recurrence plot for the Tent family for parameter $\mu = 0.65$ (a) and $\mu = 1$ (b).

changing in very irregular pattern. Similar behavior is present for the Tent family, where the blocks are even smaller in case of $\mu = 0.65$ and the dynamics variability is comparable to the Logistic family for $\mu = 1$.

6. Conclusions

In this paper, the Logistic and Tent families were introduced newly with the same parameter $\mu \in [0, 1]$ for better comparison of dynamical properties for the same fixed value of μ. For this purpose, bifurcation diagrams (Fig. 2), topological entropies (Fig. 4), Lyapunov exponents (Fig. 5), 0-1 test for chaos (Fig. 6), and recurrence matrices (Fig. 7 and Fig. 8) were computed and utilized.

As it is visible from Fig. 2(a), Fig. 4(a) and Fig. 5(a) for the value of parameters $\mu \in [0, 0.89)$, there is no chaotic behavior of the Logistic family, for this value of parameter all orbits behave like periodic or almost periodic. The crucial value $\mu = 0.8924$, also referred to as *Feigenbaum constant*, enters the region of parameters $(0.89, 1]$ for which rich irregular behavior appears. For this value of parameter chaos was detected, see Fig. 6(a). In this Fig. 6(a) the 0-1 test for chaos is also detecting "windows" visible in the bifurcation diagram Fig. 2(a), that corresponds to the constant pieces of the topological entropy observable in Fig. 4(a). The Feigenbaum constant can be effectively computed using the Newton method, see e.g. [41]. On the other hand, in Fig. 2(b), Fig. 4(b) and Fig. 5(b) for the value of parameters $\mu \in [0, 5)$ it is shown that periodic movement appears in the Tent family, the value $\mu = 0.5$ plays a key role of bifurcation border that opens a region of parameters $\mu \in [0.5, 1]$ for which chaotic patterns are recognized, see Fig. 6(b).

As it was shown and commented, T_1 and L_1 are topologically conjugated, i.e. they have the same dynamical properties. On the other hand, they have dramatically different distributions (Fig. 3) that yield computational difficulties.

Methods like topological entropy and Lyapunov exponent for detecting chaotic movements are very hard to apply for real problems since they have slow convergency property. Therefore, the 0-1 test for chaos (Fig. 6) and recurrence matrices (Fig. 7 and Fig. 8) were chosen as alternative techniques for chaos detection.

Acknowledgment

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About Authors

Marek LAMPART was born in Opava, Czech Republic. He received his M.Sc. (Mgr. and RNDr.) and Ph.D. from Mathematical Analysis in 2002 and 2005, respectively, from the Mathematical Institute in Opava, Silesian University in Opava Czech Republic. He currently holds a position of an Associate Professor at the Department of Applied Mathematics and Senior Researcher at IT4Innovations National Supercomputing Center, VSB–Technical University of Ostrava, Czech Republic. His research interests

include dynamical systems and their applications across scientific fields, mainly in economics, biological models, mechanical systems, geoinformatics, etc.

Tomas MARTINOVIC was born in Liptovsky Mikulas, Slovakia. He received his M.Sc. (Ing.) from Mathematical Methods in Economics in 2015 from Masaryk University. He is currently a doctoral student at IT4Innovations National Supercomputing Center, VSB–Technical University of Ostrava, Czech Republic. His research interests include operational research, time series analysis and dynamical systems mainly in the area of economics and road traffic.